Quantization of Forms on Cotangent Bundle

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Abstract

We consider the following construction of quantization. For a Riemannian manifold M the space of forms on T^*M is made into a space of (full) symbols of operators acting on forms on M. This gives rise to the composition of symbols, which is a deformation of the ("super")commutative multiplication of forms. The symbol calculus is exact for differential operators and the symbols that are polynomial in momenta. We calculate the symbols of natural Laplacians. (Some nice Weitzenböck like identities appear here.) Formulas for the traces corresponding to natural gradings of $\Omega(T^*M)$ are established. Using these formulas, we give a simple direct proof of the Gauss-Bonnet-Chern Theorem. We discuss these results in the connection of a general question of the quantization of forms on a Poisson manifold.

Contents

Introduction

There are two motivations for this paper.

First, for any compact Riemannian manifold M there is a natural full symbol calculus for operators acting on sections of an arbitrary vector bundle with connection. The symbols are $\operatorname{End} E$-valued functions on T^*M. This symbol calculus can be viewed as a concrete construction of quantization for the symplectic manifold T^*M. This is more or less well-known construction (see [3], [20], [6], [14], [17], [13]). Suppose now that the bundle E is the bundle of exterior differential forms. It is natural to treat forms as functions themselves (functions of odd variables) and then to go one step farther and "dequantize" the operators from $\operatorname{End} \Lambda^* M$. The new symbols obtained in this way can be identified with *forms* on symplectic manifold T^*M. This is what we do in this paper. This refined symbol calculus can be useful with the relation to the Atiyah-Singer Index Theorem. As an example, we give a straightforward proof of the Gauss-Bonnet-Chern Theorem. One of our trace formulas (formula (2.42)) can be utilized to prove the generalized Hirzebruch Signature Theorem.

The other motivation is as follows. Suppose we have a symplectic or a Poisson manifold P. At what conditions the Poisson structure can be extended from functions to differential forms ? Is it possible to quantize the algebra of forms ? In the present paper we provide a particular example of

the quantization of forms, in the case when the manifold P is a cotangent bundle T^*M.

"Supersymmetry" proofs of index theorems originated from the works of Witten and Alvarez-Gaumé [1], who used path integrals and "physical" considerations. First mathematically rigorous treatment was given for the bundle-valued Dirac operator by Getzler [6], who combined symbol calculus on Riemannian manifold with Clifford algebra. This approach was substantially simplified in [17] by an explicit use of supergeometry and deformation quantization. However, there was some peculiarity, caused by a highly asymmetric way in which even and odd variables entered the picture (see [17, p.1038-1039] and Remark 2.3 below). (Symbol calculus was abandoned in [7] and subsequent works in favor of the analysis of asymptotics of the heat kernel on the diagonal.) In this paper we work with forms instead of spinors, and this helps to achieve quantization which is more "symmetric" though leads to index formulas.

Remark (on notation). We use the standard language of supermanifolds (see [5], [10], [9], [16]). In particular, and distinguish between cotangent vectors as elements of T_x^*M and 1-forms as elements of ΠT_x^*M, where Π is the parity reversion functor. The same distinction is made between tangent vectors and "1-vectors" (multivectors of degree 1). So on even manifold vectors and covectors are even, while 1-vectors and 1-forms are odd. (As far as I know, there is no established name for the elements of ΠV for a vector space V. Borrowing some physical language, I can suggest to call them *antivectors*, and the space ΠV the *antispace*.)

Acknowledgments. The results of this paper were discussed at different time with M.A. Shubin, A. Weinstein, M. Pflaum, J. Rabin, L. Sadun, S. Rosenberg, and the participants of seminars at UC Berkeley, UCSD, University of Texas at Austin, and Boston University. N.Yu. Reshetikhin attracted my attention to the problem of quantizing forms as differential algebra. I am very much grateful to all of them.

1 Complete symbol calculus for Riemannian manifolds

Consider a Riemannian manifold M and a vector bundle E over M, with a connection. There is a natural construction that allows to assign pseu-

dodifferential operators (p.d.o.'s) to functions on T^*M. More precisely, we mean p.d.o.'s acting on the sections of E, and by functions we actually mean "End E-valued functions", i.e. the sections of $\pi^*(\operatorname{End} E)$, where $\pi : T^*M \to M$ is the projection. (The same can be done for operators $\Gamma(E) \to \Gamma(F)$ for two bundles, E, F.) In the following, the space of smooth sections of *any* bundle E over *any* manifold M is denoted by $C^\infty(M, E)$

Let us consider the geodesic segment connecting two points $x, y \in M$ (we suppose that they are sufficiently close for it to exist and be unique), and denote the point with the affine parameter s by $g_s(x, y)$. Here $g_0(x, y) = x$, $g_1(x, y) = y$. We shall need some more notation. The points of T_x^*M (cotangent vectors, or "momenta", at point $x \in M$) will be denoted by p. Their coordinates will be denoted by p_a. Momenta at the point y will be denoted by the letter q. The parallel transport $T_y^*M \to T_x^*M$ along the geodesic, w.r.t. the Levi-Civita connection, will be denoted by $T(x, y)$. The similar transport for the bundle E will be denoted by $\tau(x, y)$. We shall assume that there is a neighborhood of the diagonal Δ in $M \times M$ where the geodesic segment exists and is unique. For example, let M be compact or $M = \mathbb{R}^n$. We shall use a bump function $\alpha(x, y)$ which is identically 1 near $\Delta \subset M \times M$, nonnegative, and supported inside the indicated neighborhood of Δ. The construction depends on the choice of α, though not very heavily.

In the following we shall also need some notation for the volume elements. Liouville's measure on T^*M is denoted by $dx\,dp$ (resp., $dy\,dq$). The Riemannian volume element on M at point x is $\omega(x) = \sqrt{h(x)}\,dx$. Here $h(x) = \det(h_{ab}(x))$ is the Gram determinant in a coordinate frame. There are Euclidean volume elements on T_xM and T_x^*M. They are denoted by $\theta_x = \sqrt{g(x)}\,dv$ and $\lambda_x = (1/\sqrt{g(x)})\,dp$ respectively. Here $g(x) = \det(g_{ij}(x))$ is the Gram determinant in some chosen local frame, not necessarily coordinate. The exponential map $\exp_x : T_xM \to M$ allows to compare volume elements on M and on the tangent space at a fixed point. We introduce the function $\mu(x, y)$ by the following equation:

$$\theta_x(v) = \mu(x, y)\,\omega(y), \tag{1.1}$$

if $y = \exp_x v$, $v \in T_xM$. This definition is valid for y sufficiently close to x.

1.1 Quantization

Definition 1.1. Fix $s \in [0,1]$. Let $f \in C^\infty(T^*M, \operatorname{End} E)$. We associate with f the following operator $\hat{f}$, acting on sections of E. For $u \in C^\infty(M,E)$

$$(\hat{f}u)(x) := \frac{1}{(2\pi\hbar)^n} \int_{T^*M} dydq\; e^{\frac{i}{\hbar}(\exp_y^{-1} x)\cdot q}\; \alpha(x,y)\, \mu(x,y)$$
$$\tau(x, g_s(x,y))\; f(g_s(x,y), T(g_s(x,y),y)q)\; \tau(g_s(x,y),y)\; u(y). \quad (1.2)$$

We can make a change of variables: $y = \exp_x v, q = T(y,x)p$. This yields

Definition 1.2 (equivalent).

$$(\hat{f}u)(x) := \frac{1}{(2\pi\hbar)^n} \int_{T_xM\times T_x^*M} dvdp\; e^{-\frac{i}{\hbar}vp}\; \alpha(x, \exp_x v)$$
$$\tau(x, \exp_x sv)\; f(\exp_x sv, T(\exp_x sv, x)p)\; \tau(\exp_x sv, \exp_x v)\; u(\exp_x v). \quad (1.3)$$

These formulas might look complicated but the idea is very simple.

Example 1.1. Consider $M = \mathbb{R}^n$. Take $\alpha := 1$ (no bump function is necessary). In affine coordinates on $\mathbb{R}^n$ and in parallel frame for the (trivial) bundle E, the formula (1.2) becomes

$$(\hat{f}u)(x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} dydq\; e^{\frac{i}{\hbar}(y-x)\cdot q}\; f((1-s)x + sy), q)\; u(y), \quad (1.4)$$

and the formula (1.3) becomes

$$(\hat{f}u)(x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n\times\mathbb{R}^n} dvdp\; e^{-\frac{i}{\hbar}vp}\; f(x+sv, p)\; u(x+v). \quad (1.5)$$

For $s = 0$ this yields xp-quantization (= standard pseudodifferential calculus), for $s = 1$ this is px-quantization (the opposite way of ordering monomials), and the case $s = 1/2$ yields Weyl (symmetric) quantization (cp. [2]).

Example 1.2. Take a function $f = f(x)$, not depending on p. Then, by (1.3),

$$(\hat{f}u)(x) = \int_{T_xM} \theta_x(v)\; \delta(v)\; \tau(x, \exp_x sv)\; f(\exp_x sv)\; \tau(\exp_x sv, \exp_x v)\; u(\exp_x v)$$
$$= f(x)\, u(x). \quad (1.6)$$

(Here δ stands for the "Euclidean" delta-function on T_xM.)

1.2 Symbols

Let K be the Schwartz kernel of $A : C^\infty(M,E) \to C^\infty(M,E)$ with respect to the Riemannian volume element:

$$(Au)(x) = \int_M \omega(y)\; K(x,y)\, u(y). \tag{1.7}$$

Definition 1.3. The following function on T^*M is called the *symbol* of the operator A and is denoted by σA:

$$\begin{aligned}(\sigma A)(x,p) := \int_{T_xM} \theta_x(v)\; e^{\frac{i}{\hbar}vp}\; \alpha(\exp_x(-sv), \exp_x(1-s)v)\; \rho_s(x,v) \\ \tau(x, \exp_x(-sv))\; K(\exp_x(-sv), \exp_x(1-s)v)\; \tau(\exp_x(1-s)v, x).\end{aligned} \tag{1.8}$$

Here

$$\rho_s(x,v) := \left(\mu(\exp_x(-sv), \exp_x(1-s)v)\right)^{-1}. \tag{1.9}$$

Note that $\sigma A \in C^\infty(T^*M, \operatorname{End} E)$, and that $K(x,y) \in \operatorname{Hom}(E_y, E_x)$, for fixed x, y.

Remark 1.1. For $s = 0$ the symbol of an operator A can be calculated as follows:

$$(\sigma A)(x,p)u_0 = \left(A_y\left(e^{\frac{i}{\hbar}(\exp_x^{-1} y)\cdot p}\alpha(x,y)\; \tau(y,x)\; u_0\right)\right)_{|y:=x}, \tag{1.10}$$

for an arbitrary constant vector $u_0 \in E_x$. Here A_y means operator acting on sections which argument is denoted by y.

Theorem 1.1. *1. The quantization map $f \mapsto \hat{f}$ and the symbol map $A \mapsto \sigma A$ are "almost" mutually inverse:*

$$\sigma \hat{f} = f(1 + O(\hbar^\infty)), \tag{1.11}$$

$$K_{\widehat{\sigma A}}(x,y) = K_A(x,y) \cdot (\alpha(x,y))^2, \tag{1.12}$$

for any f and A. Here K_A denotes the Schwartz kernel of an operator A.

2. For polynomials in (p_a) and for differential operators the maps $\hat{}$ and σ are mutually inverse. In this case the construction is independent of the bump function α.

Corollary 1.1. *If A has a continuous kernel and is of trace class, then the traces of A and $\widehat{\sigma A}$ coincide.*

Proof. By equation (1.12), the kernels of A and $\widehat{\sigma A}$ coincide on the diagonal. □

Remark 1.2. A complete symbol calculus for Riemannian manifolds was for the first time considered by Bokobza-Haggiag [3] and Widom [20]. Later various modifications were suggested and used: see [6], [17], [13], [14], also [8], [4]; probably more references can be given. Though all approaches are based on the same idea (use of connection, of the exponential map and the parallel transport along geodesics), there are small subtleties leading to inequivalent constructions. In most approaches the standard pseudodifferential calculus (or "qp"-quantization) on $\mathbb{R}^n$ has been generalized. The beautiful idea to use a middle point on a geodesic segment to generalize Weyl symbols and other "s-symbols", compare [15], is due to Yu.G. Safarov. (I use the opportunity to thank M.A. Shubin from whom I learned about Safarov's approach around 1989.) The above definitions of quantization and symbols are based on this idea. For $s = 0$ the construction is basically equivalent to the one used in [17], up to a slightly different introduction of bump function. Since we follow the philosophy of deformation quantization, our formulas contain Planck constant, which is of course very important.

1.3 Composition

Definition 1.4. The operation $f \circ g := \sigma(\hat{f}\hat{g})$ is called the *composition* of functions $f, g \in C^\infty(T^*M, \operatorname{End} E)$.

Theorem 1.2. *For $\hbar \to 0$,*

$$f \circ g = fg\,(1 + O(\hbar)). \tag{1.13}$$

If f, g are scalar functions, then

$$f \circ g - g \circ f = -i\hbar\,\{f, g\}\,(1 + O(\hbar)), \tag{1.14}$$

with canonical Poisson brackets on T^*M.

Proof. The first statement follows by direct computation. To prove the second statement, note that since in the limit $\hbar \to 0$ in scalar case we obtain commutative multiplication, the commutator w.r.t. the composition induces *some* Poisson bracket on functions. Thus it is sufficient to check the brackets for local coordinates x^a, p_a. That the induced brackets for them coincide with the canonical ones, follows from the calculation below. $\square$

1.4 Examples: quantizing linear and quadratic Hamiltonians

In the subsequent calculations we fix some point $x \in M$ and work in normal coordinates centered at x. By definition, then $(\exp_x v)^a = v^a$, and $x^a = 0$, and $\Gamma^a_{bc}(0) = 0$. We also introduce a frame in E parallel along the geodesic radii $y^a = tv^a$, which we call the *parallel frame*. It is specified by two properties: $A(0) = 0$, $y^a A_a(y) = 0$ for the connection 1-form A in E. This implies that $\partial_a A_b(0)$ is antisymmetric and thus equals $F_{ab}(0)$, where F is the curvature 2-form. We can rewrite the formula (1.3) in the normal coordinates and the parallel frame:

$$(\hat{f}u)(0) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} dvdp\, e^{-\frac{i}{\hbar}vp}\, \alpha(x,v)\, f(sv, T(sv,0)p)\, u(v). \quad (1.15)$$

It looks almost as for $\mathbb{R}^n$.

Example 1.3. Let $X = X^a\partial_a \in \operatorname{Vect} M$ and consider the corresponding fiberwise-linear Hamiltonian $X \cdot \mathrm{p} = X^a p_a$. By (1.15) we have:

$$\begin{aligned}(\widehat{X \cdot \mathrm{p}}\, u)(0) &= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} dvdp\, e^{-\frac{i}{\hbar}vp}\, X^a(sv)\, (T(sv,0)p)_a\, u(v) \\ &= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} dp\, p_a \int_{\mathbb{R}^n} dv\, e^{-\frac{i}{\hbar}vp}\, (T(0,sv)X(sv))^a\, u(v) \\ &= -i\hbar \frac{\partial}{\partial v^a} \left((T(0,sv)X(sv))^a u(v)\right)_{|v=0} \\ &= -i\hbar(s \cdot \nabla_a X^a(0) + X^a(0)\, \partial_a u(0)),\end{aligned}$$

which we can rewrite in the invariant form:

$$\widehat{X \cdot \mathrm{p}} = -i\hbar\, (\nabla_X + s \operatorname{div} X). \quad (1.16)$$

If we apply this example to $X = \partial_a$, then we immediately obtain canonical commutation relations and canonical Poisson brackets.

Example 1.4. Consider the Hamiltonian $\mathrm{p}^2 = g^{ab} p_a p_b$, corresponding to the metric on M. In the same way,

$$\begin{aligned}(\widehat{\mathrm{p}^2} u)(0) &= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} dvdp \; e^{-\frac{i}{\hbar} vp} \; g^{ab}(sv) \, (T(sv,0)p)_a \, (T(sv,0)p)_b \; u(v) \\ &= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} dvdp \; e^{-\frac{i}{\hbar} vp} \; g^{ab}(0) p_a p_b \; u(v) \\ &= -\hbar^2 g^{ab}(0) \; \partial^2_{ab} u(0) = -\hbar^2 g^{ab}(0) (\nabla_a \nabla_b u - \Gamma^c_{ba} \nabla_c u)(0).\end{aligned}$$

We used the fact that parallel transport on M is an orthogonal map. Thus, independent of s,

$$\widehat{\mathrm{p}^2} = -\hbar^2 \Delta, \tag{1.17}$$

where Δ stands for the Laplace-Beltrami operator on a Riemannian manifold, acting on sections of a vector bundle with a connection:

$$\Delta u := g^{ab} (\nabla_a \nabla_b \, u - \Gamma^c_{ba} \nabla_c \, u), \tag{1.18}$$

where ∇_a in this formula denotes partial covariant derivative of the sections of E.

1.5 The trace formula

If we take as a working definition of the trace of an operator A the integral of its kernel over the diagonal $\Delta \subset M \times M$, then we easily obtain the following

Theorem 1.3. *For all $s \in [0,1]$, the trace is calculated by the same formula:*

$$\mathrm{Tr}\, \hat{f} = \frac{1}{(2\pi\hbar)^n} \int_{T^*M} dxdp \;\; \mathrm{tr}\, f(x,p), \tag{1.19}$$

$$\mathrm{Tr}\, A = \frac{1}{(2\pi\hbar)^n} \int_{T^*M} dxdp \;\; \mathrm{tr}\, (\sigma A)(x,p), \tag{1.20}$$

for any A and f. Here tr *denotes the trace on* $\mathrm{End}\, E_x$.

Proof. From (1.2) we can deduce the following formula for the restriction of the kernel of $\hat{f}$ to the diagonal:

$$K_{\hat{f}}(x,x) = \frac{1}{(2\pi\hbar)^n} \int_{T_x^*M} \lambda_x(p)\; f(x,p). \tag{1.21}$$

This is true for all s. Then (1.19) immediately follows. (Note that $\omega(x)\lambda_x(p) = dxdp$.) The equality (1.20) then follows, by Corollary 1.1. $\square$

2 Quantization of forms on T^*M

We are going to apply the preceding consideration to the particular case when $E = \Lambda(T^*M)$, the bundle of exterior differential forms. (We can also consider the case of $\Lambda(T^*M) \otimes E$, where E is some other bundle.) In this case we describe endomorphisms of $\Lambda = \Lambda_x = \Lambda(T_x^*M)$ at each x also as a result of quantization of some symbols. The corresponding symbol calculus "at a point" is a superanalog of the quantization on $\mathbb{R}^n$. The calculus of the previous section will be combined with this pointwise calculus to produce our main construction.

2.1 Quantization at a point

Consider an arbitrary local coframe on M. Its elements are considered as 1-forms and thus odd (see the remark in the introduction). So at a point x we have $n = \dim M$ odd variables $\xi^k = e^k(x)$. (Later, when we shall consider different points, we shall pick different letters for corresponding values of e^k.) Elements of Λ are functions of ξ^k. All endomorphisms of Λ are differential operators and are generated by ξ^k, $\partial/\partial\xi^k$. They also can be expressed by integral kernels:

$$(Au)(\xi) = (-1)^{n\tilde{A}} \int_{\mathbb{R}^{0|n}} \frac{D\eta}{\sqrt{g}}\; k(\xi,\eta)\; u(\eta). \tag{2.1}$$

Here $g = g(x) = \det(g_{kl}(x))$ is the Gram determinant of the frame $(e_k(x))$. Notice that

$$k_A(\xi,\eta) = (-1)^n (A\delta_\eta)(\xi), \tag{2.2}$$

where $\delta_\eta(\xi) = \sqrt{g}\,\delta(\xi - \eta)$.

Lemma 2.1.

$$k_{AB}(\xi,\eta) = (-1)^{n\tilde{A}} \int_{\mathbb{R}^{0|n}} \frac{D\xi'}{\sqrt{g}}\, k_A(\xi,\xi')\, k_B(\xi',\eta) \tag{2.3}$$

In the same way as above, or, actually, in the same way as for $\mathbb{R}^n$, we can introduce the following symbol calculus. Consider odd variables θ_k which transform contragrediently to ξ^k. That means that the form $\xi^k\theta_k$ is invariant under change of (co)frame. Geometrically, θ_k are the values at x of the elements of the dual frame e_k regarded as 1-vectors (antivectors). A function of θ_k is a multivector at the point x.

In the following we fix the point x and forget about it for time being.

Definition 2.1. Fix $r \in [0,1]$. Let $f = f(\xi,\theta)$. We associate with this function the following operator on Λ. For any $u \in \Lambda$ set

$$(\hat{f}u)(\xi) := (-i\hbar)^n \int_{\mathbb{R}^{0|2n}} D(\eta,\theta)\, e^{\frac{i}{\hbar}(\xi-\eta)\theta}\, f((1-r)\xi + r\eta, \theta)\, u(\eta). \tag{2.4}$$

Conversely, for an operator A with the kernel $k = k(\xi,\eta)$, we define its *symbol* by the formula

$$(\sigma A)(\xi,\theta) := (-1)^{n\tilde{A}} \int_{\mathbb{R}^{0|n}} \frac{D\varepsilon}{\sqrt{g}}\, e^{\frac{i}{\hbar}\varepsilon\theta}\, k(\xi - r\varepsilon, \xi + (1-r)\varepsilon). \tag{2.5}$$

Remark 2.1. For $r = 0$, we can express the symbol by the action of the operator on the exponential function (cp. (1.10):

$$(\sigma A)(\xi,\theta) = (A_\eta(e^{\frac{i}{\hbar}(\eta-\xi)\theta}))_{|\eta:=\xi}, \tag{2.6}$$

where the subscript $_\eta$ means that operator is applied to the functions of the variables η^a.

Theorem 2.1. *The quantization map $f \mapsto \hat{f}$ and the symbol map $A \mapsto \sigma A$ are mutually inverse.*

Proof. Since we deal with finite-dimensional spaces, it suffices to check either $\hat{}\circ\sigma$ or $\sigma\circ\hat{}$. Take $f = f(\xi,\theta)$. The integral kernel of $\hat{f}$, by (2.4), is

$$k_{\hat{f}}(\xi,\eta) = (-1)^{n\tilde{f}}(-i\hbar)^n \int_{\mathbb{R}^{0|n}} \sqrt{g}\, D\theta\, e^{\frac{i}{\hbar}(\xi-\eta)\theta}\, f((1-r)\xi + r\eta, \theta). \tag{2.7}$$

So, using formula (2.5),

$$
\sigma(\hat{f})(\xi,\theta) = (-1)^{n\tilde{f}} \int \frac{D\varepsilon}{\sqrt{g}}\, e^{\frac{i}{\hbar}\varepsilon\theta}\; k_{\hat{f}}(\xi - r\varepsilon, \xi + (1-r)\varepsilon)
$$

$$
= (-i\hbar)^n \int \frac{D\varepsilon}{\sqrt{g}} \sqrt{g}\, D\theta'\; e^{\frac{i}{\hbar}\varepsilon\theta}\; e^{-\frac{i}{\hbar}\varepsilon\theta'}\; f((1-r)(\xi - r\varepsilon) + r\xi + r(1-r)\varepsilon, \theta')
$$

$$
= (-i\hbar)^n \int D\varepsilon\, D\theta'\; e^{\frac{i}{\hbar}\varepsilon(\theta-\theta')}\; f(\xi,\theta') = \int D\theta'\, \delta(\theta'-\theta)\, f(\xi,\theta') = f(\xi,\theta).
$$

□

The construction of quantization is invariant under linear transformations, in the following sense. Consider two spaces $V_x, V_y \cong \mathbb{R}^{0|n}$ and an isomorphism $T: V_y \to V_x$. It induces pull-back of functions: $T^*: \Lambda_x \to \Lambda_y$. (Letters x and y are used here as just labels.)

Lemma 2.2. *For any* $f = f(\xi_x, \theta_x)$*,*

$$
T^*\; \hat{f}\; T^{*-1} = \widehat{T^* f}, \tag{2.8}
$$

where $(T^*f)(\xi_y,\theta_y) = f(T\xi_y, T^{-1}\theta_y)$*.*

Proof. Straightforward calculation, using the formula (2.4). □

(This is a trivial case of the spinor representation.) We shall use the Lemma in the next section.

The *composition* of symbols is defined in a usual way: $f \circ g := \sigma(\hat{f}\hat{g})$.

Theorem 2.2. *The composition can be calculated by the following integral formula:*

$$
(f\circ g)(\xi,\theta) =
$$

$$
(-i\hbar)^{2n} \int\limits_{\mathbb{R}^{0|2n}\times\mathbb{R}^{0|2n}} D\varepsilon_1\, D\tau_1\, D\varepsilon_2\, D\tau_2\; e^{\frac{i}{\hbar}(\varepsilon_1\tau_2 - \varepsilon_2\tau_1)}\; f(\xi + r\varepsilon_1, \theta + \tau_1)\; g(\xi + (1-r)\varepsilon_2, \theta + \tau_2) \tag{2.9}
$$

for arbitrary $r \in [0,1]$*.*
For $r = 0$ *this formula is simplified to*

$$
(f\circ g)(\xi,\theta) = (-i\hbar)^n \int_{\mathbb{R}^{0|2n}} D\varepsilon\, D\tau\; e^{-\frac{i}{\hbar}\varepsilon\tau}\; f(\xi, \theta+\tau)\; g(\xi+\varepsilon, \theta). \tag{2.10}
$$

For $r = 1$ it is simplified to

$$(f \circ g)(\xi, \theta) = (i\hbar)^n \int_{\mathbb{R}^{0|2n}} D\varepsilon\, D\tau\; e^{\frac{i}{\hbar}\varepsilon\tau}\; f(\xi + \varepsilon, \theta)\; g(\xi, \theta + \tau). \tag{2.11}$$

(Notice the difference in signs in the formulas (2.10)) and (2.11).)

Proof. To calculate the composition of f and g, one has to apply twice the formula (2.4) to find $\hat{f}\hat{g}u$ for some "test" function u, and thus obtain the integral kernel for $\hat{f}\hat{g}$, and then apply (2.5). After this straightforward calculation (that we omit), we arrive to the following formula:

$$(f \circ g)(\xi, \theta) = (-i\hbar)^{2n} \int D\varepsilon\, D\theta_2\, D\eta_1\, D\theta_1\; e^{\frac{i}{\hbar}(\varepsilon\theta + (\xi - r\varepsilon - \eta_1)\theta_1 + (\eta_1 - \xi - (1-r)\varepsilon)\theta_2)}$$
$$f((1-r)(\xi - r\varepsilon) + r\eta_1, \theta)\; g((1-r)\eta_1 + r(\xi + (1-r)\varepsilon), \theta_2). \tag{2.12}$$

Now we introduce new variables: $\varepsilon_1 = \eta_1 - (1-r)\varepsilon - \xi$, $\varepsilon_2 = \eta_1 + r\varepsilon - \xi$. This change of variables is invertible: $\eta_1 = \xi + r\varepsilon_1 + (1-r)\varepsilon_2$, $\varepsilon = \varepsilon_2 - \varepsilon_1$, and its Berezinian is unity. In these variables the formula (2.12) becomes

$$(f \circ g)(\xi, \theta) = (-i\hbar)^{2n} \int D\varepsilon_1\, D\varepsilon_2\, D\theta_1\, D\theta_2\; e^{\frac{i}{\hbar}(\varepsilon_2(\theta - \theta_1) + \varepsilon_1(\theta_2 - \theta))}$$
$$f(\xi + r\varepsilon_1, \theta_1)\; g(\xi + (1-r)\varepsilon_2, \theta_2). \tag{2.13}$$

Finally, we substitute $\theta_1 = \theta + \tau_1$, $\theta_2 = \theta + \tau_2$ and obtain (2.9). Formulas (2.10, 2.11) follow from (2.9) by direct integration. $\square$

Corollary 2.1. *For arbitrary $r \in [0, 1]$*

$$(f \circ g)(\xi, \theta) = e^{-\frac{i}{\hbar}((1-r)\frac{\partial}{\partial \theta_1}\frac{\partial}{\partial \xi_2} + r\frac{\partial}{\partial \xi_1}\frac{\partial}{\partial \theta_2})}\; f(\xi_1, \theta_1)\; g(\xi_2, \theta_2)\Big|_{\substack{\xi_1 := \xi, \quad \xi_2 := \xi \\ \theta_1 := \theta, \quad \theta_2 := \theta}} \tag{2.14}$$

Corollary 2.2. *In the limit $\hbar \to 0$ the composition becomes ordinary multiplication. For the commutator we have the following formula:*

$$f \circ g - (-1)^{\tilde{f}\tilde{g}} g \circ f = -i\hbar \,\{f, g\}\,(1 + O(\hbar)), \tag{2.15}$$

where the nonvanishing Poisson brackets for the coordinates ξ^a, θ_a are:

$$\{\theta_a, \xi^b\} = \{\xi^b, \theta_a\} = \delta_a{}^b \tag{2.16}$$

("canonical brackets").

2.2 Formulas for local traces

The space Λ can be endowed with different $\mathbb{Z}_2$-gradings. In addition to the natural grading by parity, one can define the grading by the eigenvalues of an arbitrary operator S such that $S^2 = 1$. A form u is S-*even* (resp., S-*odd*) if $Su = u$ (resp., $Su = -u$). The *trivial grading* corresponds to $S = 1$ (the identity operator). The operator S defining the natural grading is $S = (-1)^P$, where P is the "parity operator", with eigenvalues $0, 1$.

Lemma 2.3.

$$k_{(-1)^P}(\xi, \eta) = \sqrt{g}\delta(\xi + \eta) \tag{2.17}$$

Proof. $k_{(-1)^P}(\xi, \eta) = (-1)^n((-1)^P \delta_\eta)(\xi) = (-1)^n \sqrt{g}\delta(-\xi - \eta) = \sqrt{g}\delta(\xi + \eta)$ □

An operator A that preserves the S-grading is called S-*even*. This is equivalent to $AS = SA$. Similarly, A is S-*odd* if $AS = -SA$.

Notable example of $\mathbb{Z}_2$-grading is given by Hodge "star" operator. It is convenient here to present a "super" construction for the star. The Riemannian metric induces a bilinear form on the space $\mathbb{R}^{0|n} = \Pi T^*_x M$. For ξ, η denote $\xi \cdot \eta := g_{ab}\xi^a \eta^b$. We define the operator $*$ by the following formula:

$$*u(\xi) := \text{const} \int_{\mathbb{R}^{0|n}} \frac{D\eta}{\sqrt{g}}\, e^{-it\,\xi\cdot\eta}\, u(\eta). \tag{2.18}$$

Here t is an arbitrary parameter, and const stands for a normalizing factor, which can be chosen by convenience.

Lemma 2.4. *In components, if* $u = u_{a_1,\ldots,a_k}\, \xi^{a_1} \ldots \xi^{a_k}$, *where we assume the coefficients skew-symmetric,*

$$(*u)_{a_1,\ldots,a_{n-k}} = \text{const}\ t^{n-k}\, i^{(n-k)^2+n(n-1)} \frac{\sqrt{g}}{(n-k)!}\, u^{b_1,\ldots,b_k}\ \varepsilon_{b_1,\ldots,b_k,a_1,\ldots,a_{n-k}}. \tag{2.19}$$

Here the indices are raised with the help of the metric, and $\varepsilon_{b_1,\ldots,b_k,a_1,\ldots,a_{n-k}}$ *is the standard Levi-Civita symbol. So the operation* $*$ *defined by (2.18) differs from usual only by some factor, which depends on the degree of the form. For the inverse operator the following formula:*

$$*^{-1} = (\text{const})^{-2}\,(it)^{-n}(-1)^{n(n-1)/2}\ * \tag{2.20}$$

holds for any n.

Consider the case $n = 2m$. Then the operator $*$ is even (in the sense of natural grading). Set const $:= t^{-m}$. By Lemma 2.4, $*^2 = 1$. So we can consider $*$-grading. $*$-Even forms are what is called "self-dual", and $*$-odd are "anti self-dual".

To each choice of $\mathbb{Z}_2$-grading corresponds the *trace*, defined by the formula:

$$\operatorname{tr}_S A := \operatorname{tr} A_{00} - A_{11} \tag{2.21}$$

for an S-even operator A, where the block decomposition corresponds to the chosen grading. Up to a factor, tr_S is specified by the property that it annihilates "S-commutators" (commutators of operators w.r.t. the chosen $\mathbb{Z}_2$-grading).

Let us consider the traces corresponding to parity, trivial grading, and $*$-grading (the last one only for even-dimensional case). Denote them by str, tr_1 and tr_* respectively. Obviously, $\operatorname{tr}_S A = \operatorname{tr}_1(AS) = \operatorname{tr}_1(SA)$ for any S.

Theorem 2.3. *For any operator A,*

$$\operatorname{str} A = (-1)^{\tilde{A}n} \int\limits_{\mathbb{R}^{0|n}} \frac{D\xi}{\sqrt{g}}\, k_A(\xi,\xi) = (-i\hbar)^n \int\limits_{\mathbb{R}^{0|2n}} D(\xi,\theta)\, \sigma A(\xi,\theta) \tag{2.22}$$

(independent of r),

$$\operatorname{tr}_1 A = (-1)^{n\tilde{A}} \int\limits_{\mathbb{R}^{0|n}} \frac{D\xi}{\sqrt{g}}\, k_A(-\xi,\xi) = (-i\hbar)^n \int\limits_{\mathbb{R}^{0|2n}} D(\xi,\theta)\;\, e^{-\frac{2i}{\hbar}\xi\theta}\, \sigma A((2r-1)\xi,\theta), \tag{2.23}$$

and

$$\operatorname{tr}_* A = t^{-m} \int\limits_{\mathbb{R}^{0|2m}\times\mathbb{R}^{0|2m}} \frac{D\xi D\eta}{g}\, e^{-it\,\xi\cdot\eta}\, k_A(\xi,\eta) = t^{-m} \int\limits_{\mathbb{R}^{0|2m}} \frac{D\xi}{\sqrt{g}}\, \sigma A(\xi, t\hbar\,\xi_{\#}) \tag{2.24}$$

(independent of r), where $\xi_{\#a} = g_{ab}\xi^b$. The last formula works for $n = 2m$.

Proof. Let us prove the first equality in (2.22). Consider $[A,B] = AB - (-1)^{\tilde{A}\tilde{B}}BA$. Applying Lemma 2.1 and restricting the kernel to to the diagonal, one can deduce

$$k_{[A,B]}(\xi,\xi) = \int \frac{D\xi'}{\sqrt{g}}\, (-1)^{n\tilde{A}}\, (k_A(\xi,\xi')\, k_B(\xi',\xi) - (-1)^n\, (k_A(\xi',\xi)\, k_B(\xi,\xi')),$$

which obviously gives zero after the integration w.r.t. ξ. Thus the right hand side of our formula annihilates commutators, and hence it is proportional to the trace str. To obtain the normalization factor, it suffices to check some particular operator. Consider $A : u \mapsto u(0)$. For it, $\operatorname{str} A = 1$ and $k_A(\xi,\eta) = \sqrt{g}\,\eta^n \dots \eta^1$, so

$$\int \frac{D\xi}{\sqrt{g}}\, k_A(\xi,\xi) = \int D\xi\; \xi^n \dots \xi^1 = 1 = \operatorname{str} A.$$

Thus the normalization factor actually equals 1. The expression for the trace via the symbol follows from the equality

$$k_{\hat{f}}(\xi,\xi) = (-1)^{n\tilde{f}}(-i\hbar)^n \int_{\mathbb{R}^{0|n}} \sqrt{g}\, D\theta\; f(\xi,\theta) \tag{2.25}$$

(see (2.7)). So the formula (2.22) is completely proved. To prove (2.23), notice that $\operatorname{tr}_1 A = \operatorname{str}((-1)^P A) = \operatorname{str}(A(-1)^P)$. By Lemmas 2.1 and 2.3,

$$k_{(-1)^P A}(\xi,\eta) = \int \frac{D\xi'}{\sqrt{g}}\, \sqrt{g}\delta(\xi+\xi')\, k_A(\xi',\eta) = k_A(-\xi,\eta).$$

Applying the formula for str, we obtain the first equality in (2.23). Expressing $k_A(-\xi,\eta)$ via the symbol completes the proof of (2.23). Finally, to obtain (2.24), we apply (2.23) to the operator $*A$. From Lemma 2.1 and the formula (2.18), we obtain

$$k_{*A}(-\xi,\xi) = \int \frac{D\xi'}{\sqrt{g}}\, t^{-m}\; e^{it\,\xi\cdot\xi'}\; k_A(\xi',\xi).$$

Thus,

$$\operatorname{tr}_1(*A) = \int \frac{D\xi\, D\eta}{g}\, t^{-m}\; e^{-it\xi\cdot\eta}\; k_A(\xi,\eta) =$$
$$\int \frac{D\xi\, D\eta}{g}\, t^{-m}\; e^{-it\xi\cdot\eta} \int (-i\hbar)^{2m} \sqrt{g}\, D\theta\; e^{\frac{i}{\hbar}(\xi-\eta)\theta}\; f((1-r)\xi + r\eta, \theta).$$

Substitute $\xi = \varepsilon - r\zeta$, $\eta = \varepsilon + (1-r)\zeta$. The Berezinian of this change of variables is unity. Now, $(1-r)\xi + r\eta = \varepsilon$, $\xi - \eta = -\zeta$, and the argument of the first exponential becomes $-it(\varepsilon - r\zeta)\cdot(\varepsilon + (1-r)\zeta) = -it(\varepsilon\cdot(1-r)\zeta - r\zeta\cdot\varepsilon) =$

$it\zeta\cdot\varepsilon$. So,

$$\operatorname{tr}_* A = \int \frac{D\varepsilon\, D\theta\, D\zeta}{\sqrt{g}}\, t^{-m}(-i\hbar)^{2m}\, e^{it\zeta\cdot\varepsilon - \frac{i}{\hbar}\zeta\theta}\, f(\varepsilon,\theta) =$$
$$\int \frac{D\varepsilon\, D\theta\, D\zeta}{\sqrt{g}}\, t^{-m}(-i\hbar)^{2m}\, e^{-\frac{i}{\hbar}\zeta(\theta - t\hbar\varepsilon_{\#})}\, f(\varepsilon,\theta) =$$
$$\int \frac{D\varepsilon\, D\theta}{\sqrt{g}}\, t^{-m}\, \delta(\theta - t\hbar\,\varepsilon_{\#})\, f(\varepsilon,\theta) = \int \frac{D\varepsilon}{\sqrt{g}}\, t^{-m}\, f(\varepsilon, t\hbar\,\varepsilon_{\#}),$$

and the theorem is completely proved. □

Remark 2.2. For $r = 1/2$ (the Weyl quantization), it follows from the formula (2.23) that $\operatorname{tr}_1 A = 2^n\,(\sigma A)(0,0)$.

2.3 Global quantization

Now we can combine the quantization built in Section 1 with the above "fiberwise" quantization.

We describe forms on M as functions on the supermanifold $\hat{M} = \Pi TM$. Locally they look as functions $u = u(x,\xi)$, where ξ^a are elements of a coframe at x. For the coordinate coframe, $\xi^a = dx^a$. Operators on forms can be described by "Schwartz kernels" of the appearance $K(x,\xi,y,\eta)$:

$$(Au)(x,\xi) = (-1)^{n\tilde{A}} \int_{\hat{M}} D(y,\eta)\; K(x,\xi,y,\eta)\; u(y,\eta), \tag{2.26}$$

for $\xi^a = dx^a$. In the coordinate-free language, the kernel is a generalized form (de Rham's current) on $M \times M$, and $Au = \pi_{2*}(K\pi_1^* u)$, where $\pi_1, \pi_2 : M \times M \to M$ are projections.

After certain simplifications, we obtain the following formulas. Below the letter T stands for the parallel transport along geodesics, w.r.t. the Levi-Civita connection, in the bundles T^*M, ΠTM and $T^*M\Pi$.

For quantization. Suppose we have a function $f = f(x,p,\xi,\theta)$. Geometrically, this is a section of the pull-back $\pi^*(\Lambda(TM \oplus T^*M))$, by $\pi : T^*M \to M$, or a function on the supermanifold $T^*M \oplus \Pi TM \oplus T^*M\Pi)$. Denote this supermanifold by N (this is a bundle over M). To this function we assign the

following operator, acting on $\Omega(M)$:

$$(\hat{f}u)(x,\xi) := \frac{1}{(2\pi i)^n}\int_N D(y,q,\eta,\theta)\ \alpha(x,y)\mu(x,y)\ e^{\frac{i}{\hbar}\left(\exp_y^{-1}x\cdot q+(T(y,x)\xi-\eta)\theta\right)}$$
$$f(g_s(x,y), T(g_s(x,y),y)q, (1-r)\,T(g_s(x,y),x)\xi + r\,T(g_s(x,y),y)\eta,$$
$$T(g_s(x,y),y)\theta)\ \ u(y,\eta). \quad (2.27)$$

Equivalent definition:

$$(\hat{f}u)(x,\xi) := \frac{1}{(2\pi i)^n}\int D(v,p,\varepsilon,\theta)\alpha(x,\exp_x v)\ e^{-\frac{i}{\hbar}(vp+\varepsilon\theta)}$$
$$f(\exp_x sv, T(\exp_x sv,x)p, T(\exp_x sv,x)(\xi+r\varepsilon), T(\exp_x sv,x)\theta)$$
$$u(\exp_x v, T(\exp_x v,x)(\xi+\varepsilon)). \quad (2.28)$$

Here integration is over the fiber $T_xM \oplus T_x^*M \oplus \Pi T_xM \oplus T_x^*M\Pi)$.

For symbols. Let A be an operator on differential forms, with the Schwartz kernel $K(x,\xi,y,\eta)$. (Here ξ^a are elements of a coframe at x, and η^a are elements of a coframe at y.) Then its *symbol* $\sigma A = (\sigma A)\,(x,p,\xi,\theta)$ is defined by the formula

$$(\sigma A)(x,p,\xi,\theta) := \int_{T_xM\times\Pi T_xM} D(v,\varepsilon)\ \alpha(\exp_x(-sv),\exp_x(1-s)v)\ \rho_s(x,v)$$
$$e^{\frac{i}{\hbar}(vp+\varepsilon\theta)}\ \ K(\exp_x(-sv), T(\exp_x(-sv),x)(\xi-r\varepsilon), \exp_x(1-s)v,$$
$$T(\exp_x(1-s)v,x)(\xi+(1-r)\varepsilon)). \quad (2.29)$$

These formulas can be obtained from (1.2, 1.8) and (2.4, 2.5) by more or less direct calculation. At a certain step it is necessary to use Lemma 2.2 to change the points at which reference frames are taken, and to change variables in the integrals.

Our constructions depend on two real parameters $s,r\in[0,1]$. Note also that without any difficulties the definitions of quantization and symbols can be extended from "scalar-valued" forms to forms taking values in an arbitrary vector bundle with a connection. Quite helpful for practical calculations is the appearance of the formula (2.28) in normal coordinates centered at x (so $x^a=0$, $(\exp_x v)^a = v^a$):

$$(\hat{f}u)(0,\xi) := \frac{1}{(2\pi i)^n}\int D(v,p,\varepsilon,\theta)\alpha(x,v)\ e^{-\frac{i}{\hbar}(vp+\varepsilon\theta)}$$
$$f(sv, T(sv,0)p, T(sv,0)(\xi+r\varepsilon), T(sv,0)\theta)\ u(v,T(v,0)(\xi+\varepsilon)). \quad (2.30)$$

As it was said, the symbols in our refined symbol calculus are functions on the supermanifold $N = T^*M \oplus \Pi TM \oplus T^*M\Pi$. Roughly speaking, they are tensor products of forms and multivectors on M, depending also on a point p in the cotangent space. Luckily, they can be given a nicer description. Using the Levi-Civita connection on M, the supermanifold N can be identified with $\widehat{T^*M}$. That means that our symbols can be considered simply as forms on T^*M. For clarity, let us write down the change of coordinates on N (assuming that we use coordinate frames):

$$\begin{cases} x^a = x^a(x') \\ p_a = \dfrac{\partial x^{a'}}{\partial x^a}(x(x'))\, p_{a'} \\ dx^a = dx^{a'}\, \dfrac{\partial x^a}{\partial x^{a'}}(x') \\ \theta_a = \dfrac{\partial x^{a'}}{\partial x^a}(x(x'))\, \theta_{a'} \end{cases} \tag{2.31}$$

Compare it with $\widehat{T^*M}$:

$$\begin{cases} x^a = x^a(x') \\ p_a = \dfrac{\partial x^{a'}}{\partial x^a}(x(x'))\, p_{a'} \\ dx^a = dx^{a'}\, \dfrac{\partial x^a}{\partial x^{a'}}(x') \\ dp_a = d\left(\dfrac{\partial x^{a'}}{\partial x^a}(x(x'))\right) p_{a'} + \dfrac{\partial x^{a'}}{\partial x^a}(x(x'))\, dp_{a'} \end{cases} \tag{2.32}$$

The tempting idea to identify θ_a with dp_a fails because of the additional term in the transformation law for dp_a. However, using the connection, we can identify θ_a with ∇p_a, where

$$\nabla p_a = dp_a - dx^b\, \Gamma^c_{ab}\, p_c. \tag{2.33}$$

Notice that the change of variables from (x, p, dx, dp) to $(x, p, dx, \nabla p)$ has the unit Berezinian, hence all our integrals over (x, p, ξ, θ) can be rewritten as integrals over (x, p, dx, dp), i.e., as integrals of differential forms over the manifold T^*M.

Remark 2.3. In the paper [6] Getzler proposed a complete symbol calculus for spinor fields, combining a symbol calculus on manifolds with the "Weyl" isomorphism $\Lambda(V) \to C(V)$ (where $\Lambda(V)$ and $C(V)$ are respectively exterior and Clifford algebra of a Euclidean space V). The resulting symbols were *horizontal* forms on T^*M, i.e., functions $f = f(x, p, \xi)$ in our notation. The key element of that construction was some nonobvious filtration introduced into the algebra of symbols (by the total degree in p_a and ξ^b, in our notation). As we showed in [17], in the quantization language this was equivalent to a very peculiar convention $\hbar_{\text{spinor}} = \hbar^2_{\text{usual}}$ (in [6] no Planck constant was used, so this fact was not explicit there). That strange convention was nevertheless essential for calculating the Dirac index, see [17]. Comparing with the complete symbol calculus constructed it this section, we see that now *all* forms on T^*M appear as symbols, and odd and even variables are on equal footing (no discrimination w.r.t. $\hbar$).

2.4 Examples

The exterior differential and the codifferential (or divergence), which is defined on Riemannian manifold, are given by the formulas:

$$d = dx^a \frac{\partial}{\partial x^a} = \xi^a \nabla_a \tag{2.34}$$

$$\delta = g^{ab} \frac{\partial}{\partial \xi^a} \nabla_b. \tag{2.35}$$

(Notice that the second expression for d in (2.34) is valid for arbitrary frame.) The covariant derivative of (inhomogeneous) form is

$$\nabla_a = \frac{\partial}{\partial x^a} - \Gamma_{ak}{}^l \xi^k \frac{\partial}{\partial \xi^l}, \tag{2.36}$$

where $\Gamma_{ak}{}^l$ is the Christoffel symbol.

Remark 2.4. In components: if $u = u_{a_1,\dots,a_k} dx^{a_1} \dots dx^{a_k}$, then

$$(\delta u)_{a_1,\dots,a_{k-1}} = k\, u_{aa_1,\dots,a_{k-1}}{}^{;a}. \tag{2.37}$$

Direct application of the formula (2.30) gives the following results for the quantization of symbols $\xi \mathrm{p} = \xi^a p_a$ and $\theta \cdot \mathrm{p} = g^{ab}\theta_a p_b$:

Example 2.1.

$$\widehat{\xi \mathrm{p}} = -i\hbar\, d \tag{2.38}$$

Example 2.2.

$$\widehat{\theta \cdot \mathrm{p}} = -\hbar^2\, \delta \tag{2.39}$$

2.5 Traces

Theorem 2.4. *For any operator A acting on $\Omega(M)$,*

$$\operatorname{Str} A = \frac{1}{(2\pi i)^n} \int_N D(x,p,\xi,\theta)\, \sigma A(x,p,\xi,\theta) = \frac{i^n}{(2\pi)^n} \int_{T^*M} \sigma A \tag{2.40}$$

(integral of differential form over T^*M), independent of s and r,

$$\operatorname{Tr}_1 A = \frac{1}{(2\pi i)^n} \int_N D(x,p,\xi,\theta)\ \ e^{-\frac{2i}{\hbar}\xi\theta} \sigma A(x,p,(2r-1)\xi,\theta), \tag{2.41}$$

independent of s. For $n = 2m$, and for any A,

$$\operatorname{Tr}_* A = \frac{1}{(2\pi)^{2m}} \int \frac{D(x,p,\xi)}{\sqrt{g}}\, \sigma A(x,p,\xi,\hbar^{-1}\,\xi_\#), \tag{2.42}$$

independent of s and r. Here $(\xi_\#)_a = g_{ab}\xi^b$. (We set in the last formula $t := \hbar^{-2}$.)

Proof. Immediately follows from Theorem 1.3 and Theorem 2.3. ∎

Notice that these formulas contain no powers of $\hbar$ in front of the integrals, compared to (1.20) and (2.22, 2.23, 2.24). This comes naturally by cancellation of the powers of $\hbar$ from the "even" and the "odd" parts of our quantization.

3 Natural Laplacians and their symbols

3.1 Weitzenböck formula

For any vector bundle with a connection over Riemannian M there is an operator

$$\Delta = g^{ab}(\nabla_a \nabla_b - \Gamma_{ab}{}^c \nabla_c) \tag{3.1}$$

(here ∇ stands for the covariant derivative of the sections of the bundle). This Laplacian is called Laplace-Beltrami operator or Bochner Laplacian. In the case when the bundle is the exterior bundle $\Lambda M = \Lambda(T^*M)$, there is another natural construction of a Laplacian operator, namely the Hodge Laplacian $\Box = (d+\delta)^2 = d\delta + \delta d$. The relation between these two operators is given by the "Weitzenböck formula" [19, p. 397] (see also [11]). (This name is used, in general, for formulas relating any natural Laplacian operators in arbitrary bundle; by a "Laplacian operator" is meant an operator generalizing the usual Laplacian on functions in the Euclidean space, in the sense that in components it's the usual Laplacian applied componentwise, plus some lower-order terms.) We shall use it in the following form.

Theorem 3.1.

$$\Box = (d+\delta)^2 = \Delta + \mathrm{Ric}_a{}^b\, dx^a \frac{\partial}{\partial dx^b} + R_a{}^k{}_b{}^l\, dx^a dx^b \frac{\partial}{\partial dx^k}\frac{\partial}{\partial dx^l} \tag{3.2}$$

$$= \Delta + \mathrm{Ric}_a{}^b\, dx^a \frac{\partial}{\partial dx^b} + \frac{1}{2} R_{ab}{}^{kl}\, dx^a dx^b \frac{\partial}{\partial dx^k}\frac{\partial}{\partial dx^l}, \tag{3.3}$$

where $R_{ab}{}^{kl}$ is the Riemann tensor, $\mathrm{Ric}_a{}^b = R_{ka}{}^{kb}$ is the Ricci tensor, indices are raised and lowered by the Riemannian metric.

Proof. By the formulas (2.34) and (2.35), $d+\delta = \hat\gamma^a\, \nabla_a$, where we define $\hat\gamma^a := \xi^a + g^{ak}\partial/\partial\xi^k$. We need to find $(d+\delta)^2 = (1/2)\,[d+\delta, d+\delta]$. Directly:

$$\begin{aligned}[d+\delta, d+\delta] &= [\hat\gamma^a\, \nabla_a, \hat\gamma^b\, \nabla_b] \\ &= \hat\gamma^a\, [\nabla_a, \hat\gamma^a]\, \nabla_b + [\hat\gamma^a, \hat\gamma^b]\, \nabla_a \nabla_b - \hat\gamma^b \hat\gamma^a\, [\nabla_a, \nabla_b] - \hat\gamma^b\, [\hat\gamma^a, \nabla_b]\, \nabla_a. \end{aligned} \tag{3.4}$$

Observe the following facts. First, $[\hat\gamma^a, \hat\gamma^b] = 2g^{ab}$. Second, $[\nabla_a, \nabla_b] = -R_{abk}{}^l \xi^k \partial/\partial\xi^l$ (cp.(2.36)). Third, in the frame associated with normal coordinates centered at a given point x_0, the operators $\hat\gamma^a$ and ∇_b commute at x_0 (since the Christoffel symbols and partial derivatives of the metric vanish at x_0). Let's calculate in these coordinates. Thus, at x_0,

$$[d+\delta, d+\delta] = g^{ab}\, \nabla_a \nabla_b + \hat\gamma^b \hat\gamma^a\, R_{abp}{}^q \xi^p \frac{\partial}{\partial \xi^q} = 2\Delta + \hat\gamma^a \hat\gamma^b\, R_{abp}{}^q \xi^p \frac{\partial}{\partial \xi^q}.$$

Since (by direct computation)

$$\hat\gamma^b\hat\gamma^a = \xi^b\xi^a + g^{ba} + (\xi^b g^{ak} - \xi^a g^{bk})\frac{\partial}{\partial\xi^k} + g^{bl}g^{ak}\frac{\partial}{\partial\xi^l}\frac{\partial}{\partial\xi^k},$$

then, using the properties of the Riemann tensor,

$$\begin{aligned}[]
[d+\delta, d+\delta] &= 2\Delta - 2\xi^a g^{bk}\, R_{abp}{}^q \frac{\partial}{\partial\xi^k}\xi^p\frac{\partial}{\partial\xi^q} = 2\Delta - 2R_a{}^k{}_p{}^q\, \xi^a\frac{\partial}{\partial\xi^k}\xi^p\frac{\partial}{\partial\xi^q} \\
&= 2\Delta + 2R_a{}^k{}_p{}^q\,(\xi^a\xi^p\frac{\partial}{\partial\xi^k}\frac{\partial}{\partial\xi^q} - \delta_k{}^p\xi^a\frac{\partial}{\partial\xi^q}) \\
&= 2\left(\Delta + R_a{}^k{}_b{}^l\,\xi^a\xi^b\frac{\partial}{\partial\xi^k}\frac{\partial}{\partial\xi^l} + \mathrm{Ric}_a{}^b\xi^a\frac{\partial}{\partial\xi^b}\right),
\end{aligned}$$

and the formula (3.2) is proved. To deduce (3.3), one should raise all indices in $R_a{}^k{}_b{}^l$ (respectively lowering them for $\xi^a\xi^b$), alternate in a, b (the factor 1/2 appears) and apply the Ricci identity $R^{akbl} + R^{bakl} + \underbrace{R^{kbal}}_{-R^{bkal}} = 0$. □

Remark 3.1. Presentation of $d+\delta$ as a "Dirac" operator which we used in the proof of Theorem 3.1 corresponds to the action on forms of the Clifford algebra $C(n)$, induced by the Riemannian metric. Operators $\hat\gamma^a$ *do not* generate all endomorphisms of Λ. At the same time, the quantization that we constructed in the previous subsection actually realizes forms as spinors for larger Clifford algebra $C(n,n) \cong \mathrm{End}\,\Lambda$, which is independent of metric. So there are two different Clifford module structures on the space Λ, for two different Clifford algebras, which should not lead to a confusion.

Remark 3.2. Formulas (3.2, 3.3) can be also rewritten as $\Box = \Delta + R$, with the operator $R = R_a{}^k{}_b{}^l\xi^a\frac{\partial}{\partial\xi^k}\xi^b\frac{\partial}{\partial\xi^l}$, see [12]. In such form the Weitzenböck formula was used in [1]. The "normal" form (3.3) is most suitable for our purposes.

3.2 The symbol of the Hodge Laplacian

Theorem 3.2. *For any $s \in [0,1]$,*

$$\begin{aligned}
&(\sigma(\Box))\,(x,p,\xi,\theta) = \\
&= -\hbar^{-2}\left(\mathrm{p}^2 + \frac{1}{2}\,R_{ab}{}^{kl}\,\xi^a\xi^b\theta_k\theta_l\right) + i\hbar^{-1}\,(1-2r)\;\mathrm{Ric}_a{}^b\,\xi^a\theta_b + r(1-r)R
\end{aligned} \tag{3.5}$$

Proof. We apply the Weitzenböck formula (3.3). Example 1.17 provides $\sigma(\Delta) = -\hbar^{-2}\,\mathrm{p}^2$, independent of s. All we need, is to calculate the symbols of the remaining terms. Denote $A := \mathrm{Ric}_a{}^b\,\xi^a\partial/\partial\xi^b$, $B := R_{ab}{}^{kl}\,\xi^a\xi^b\partial/\partial\xi^k\partial/\partial\xi^l$. For simplicity we shall write below $R_a{}^b$ instead of $\mathrm{Ric}_a{}^b$. First we find the kernels:

$$
\begin{aligned}
k_A(\xi,\eta) &= R_a{}^b\xi^a\,\frac{\partial}{\partial\xi^b}\,\delta(\eta-\xi)\,\sqrt{g},\\
k_B(\xi,\eta) &= R_{ab}{}^{kl}\,\xi^a\xi^b\frac{\partial}{\partial\xi^k}\frac{\partial}{\partial\xi^l}\,\delta(\eta-\xi)\,\sqrt{g}.
\end{aligned}
$$

Now we change variables: $\xi = \zeta - r\varepsilon, \eta = \zeta + (1-r)\varepsilon$, or $\varepsilon = \eta - \xi, \zeta = (1-r)\xi + r\eta$. As $\partial/\partial\xi = (1-r)\partial/\partial\zeta - \partial/\partial\varepsilon$, it follows that

$$
k_A(\zeta - r\varepsilon, \zeta + (1-r)\varepsilon) = -R_a{}^b(\zeta^a - r\varepsilon^a)\frac{\partial}{\partial\varepsilon^b}\delta(\varepsilon)\,\sqrt{g}, \tag{3.6}
$$

$$
k_B(\zeta - r\varepsilon, \zeta + (1-r)\varepsilon) = R_{ab}{}^{kl}(\zeta^a - r\varepsilon^a)(\zeta^b - r\varepsilon^b)\frac{\partial^2}{\partial\varepsilon^k\partial\varepsilon^l}\delta(\varepsilon)\,\sqrt{g}. \tag{3.7}
$$

Substituting (3.6) and (3.7) into (2.5) and integrating by parts, we obtain

$$
\begin{aligned}
&(\sigma A)(\xi,\theta) = \int D\varepsilon\,\delta(\varepsilon)\,\frac{\partial}{\partial\varepsilon^b}\left(-R_a{}^b e^{\frac{i}{\hbar}\varepsilon\theta}(\xi^a - r\varepsilon^a)\right)\\
&= \int D\varepsilon\,\delta(\varepsilon)\,\left(-R_a{}^b\,\frac{i}{\hbar}\,\theta_b\,e^{\frac{i}{\hbar}\varepsilon\theta}(\xi^a - r\varepsilon^a) + R_a{}^b\,e^{\frac{i}{\hbar}\varepsilon\theta}\,r\delta_b{}^a\right) = \frac{i}{\hbar}\,R_a{}^b\,\xi^a\theta_b + rR,\\
&(\sigma B)(\xi,\theta) = \int D\varepsilon\,\delta(\varepsilon)\,\frac{\partial^2}{\partial\varepsilon^k\partial\varepsilon^l}\left(R_{ab}{}^{kl}e^{\frac{i}{\hbar}\varepsilon\theta}(\xi^a - r\varepsilon^a)(\xi^b - r\varepsilon^b)\right)\\
&= \int D\varepsilon\,\delta(\varepsilon)\,\frac{\partial}{\partial\varepsilon^k}\left(\frac{i}{\hbar}\,\theta_l\,e^{\frac{i}{\hbar}\varepsilon\theta}\,R_{ab}{}^{kl}\,(\xi^a - r\varepsilon^a)(\xi^b - r\varepsilon^b) + e^{\frac{i}{\hbar}\varepsilon\theta}\,2R_{lb}{}^{kl}(-r)(\xi^b - r\varepsilon^b)\right)\\
&= \int D\varepsilon\,\delta(\varepsilon)\,\Big(-\Big(\frac{i}{\hbar}\Big)^2\,\theta_l\theta_k\,e^{\frac{i}{\hbar}\varepsilon\theta}\,R_{ab}{}^{kl}\,(\xi^a - r\varepsilon^a)(\xi^b - r\varepsilon^b) - \frac{i}{\hbar}\,\theta_l\,e^{\frac{i}{\hbar}\varepsilon\theta}\,2R_{kb}{}^{kl}\cdot\\
&\quad(-r)(\xi^b - r\varepsilon^b) + \frac{i}{\hbar}\,\theta_k\,e^{\frac{i}{\hbar}\varepsilon\theta}\,2R_{lb}{}^{kl}(-r)(\xi^b - r\varepsilon^b) + e^{\frac{i}{\hbar}\varepsilon\theta}\,2R_{lk}{}^{kl}(-r)(-r)\Big)\\
&\quad= \Big(\frac{i}{\hbar}\Big)^2 R_{ab}{}^{kl}\,\xi^a\xi^b\theta_k\theta_l + \frac{i}{\hbar}\left(-r2R_{kb}{}^{kl}\,\xi^b\theta_l + r2R_{lb}{}^{kl}\,\xi^b\theta_k\right) + r^2\,2R_{lk}{}^{kl}\\
&\qquad\qquad= \Big(\frac{i}{\hbar}\Big)^2 R_{ab}{}^{kl}\,\xi^a\xi^b\theta_k\theta_l + \frac{i}{\hbar}(-4r)\,R_a{}^k\xi^a\theta_k - 2r^2R.
\end{aligned}
$$

(where we restored ξ in the argument). Finally we obtain

$$\sigma\Box = \sigma\Delta + \sigma A + \frac{1}{2}\sigma B$$
$$= \sigma\Delta + i\hbar^{-1}R_a{}^b\xi^a\theta_b + rR + -\hbar^2\frac{1}{2}R_{ab}{}^{kl}\,\xi^a\xi^b\theta_k\theta_l + i\hbar^{-1}(-2r)R_a{}^b\xi^a\theta_b - r^2R$$
$$= -\hbar^2\left(\mathrm{p}^2 + \frac{1}{2}R_{ab}{}^{kl}\,\xi^a\xi^b\theta_k\theta_l\right) + i\hbar^{-1}(1-2r)R_a{}^b\xi^a\theta_b + r(1-r)R,$$

which completes the proof. □

4 Gauss-Bonnet-Chern Theorem

As it is very well known, the index of any elliptic operator $A : \Gamma(M, E_0) \to \Gamma(M, E_1)$ on compact manifold M can be expressed as the supertrace of a suitable function of the corresponding "Laplacian" $\Delta_A = AA^* + A^*A : \Gamma(M, E_0 \oplus E_1) \to \Gamma(M, E_0 \oplus E_1)$:

$$\operatorname{index} A = \operatorname{str}\varphi(\Delta_A), \tag{4.1}$$

provided the trace is absolutely convergent, and $\varphi(0) = 1$. In particular,

$$\operatorname{index} A = \operatorname{str} e^{t\Delta_A}, \tag{4.2}$$

for suitable t. In this section we shall apply this formula to obtain the expression for the Euler characteristic of the Riemannian manifold M via curvature, i.e., to deduce the Gauss-Bonnet-Chern Theorem. Our main tool will be the formula for the supertrace (2.40).

Consider $D = d + \delta : \Omega^{ev}(M) \to \Omega^{od}(M)$. We have

$$\chi(M) = \operatorname{index} D = \operatorname{str} e^{t\Box} \tag{4.3}$$

for suitable t. Since

$$\sigma e^{t\Box} = e^{t\sigma(\Box)}\,(1 + O(\hbar)) = e^{t(-\hbar^{-2}(\mathrm{p}^2+\dots)(1+O(\hbar))}\,(1 + O(\hbar)) \tag{4.4}$$

(see (3.5)), $t = \hbar^2$ seems a good choice. So

$$\sigma(\hbar^2\Box) = -\mathrm{p}^2 - \frac{1}{2}\,R_{ab}{}^{kl}\,\xi^a\xi^b\theta_k\theta_l + O(\hbar), \tag{4.5}$$

and

$$\sigma e^{\hbar^2\Box} = e^{\sigma(\hbar^2\Box)}\,(1+O(\hbar)) = e^{-\mathrm{p}^2-\frac{1}{2}\,R_{ab}{}^{kl}\,\xi^a\xi^b\theta_k\theta_l} + O(\hbar). \tag{4.6}$$

Passing to the limit $\hbar \to 0$ (which is harmless since the supertrace (4.3) is independent of $\hbar$), we obtain:

$$\begin{aligned}
\chi(M) = \operatorname{index} D = \operatorname{str} e^{\hbar^2\Box} &= \frac{1}{(2\pi i)^n}\int_N D(x,p,\xi,\theta)\; e^{-\mathrm{p}^2-\frac{1}{2}\,R_{ab}{}^{kl}\,\xi^a\xi^b\theta_k\theta_l} \\
&= \begin{cases} 0, & n=2m+1 \\ \dfrac{(\sqrt{\pi})^{2m}}{(2\pi i)^{2m}}\displaystyle\int\; D(x,\xi)\;\operatorname{Pf}(R_{ab}{}^{kl}\,\xi^a\xi^b), & n=2m \end{cases} \\
= \begin{cases} 0, \quad n=2m+1 \\ (-1)^m\displaystyle\int D(x,\xi)\;\operatorname{Pf}\Big(\frac{1}{2\pi}\;\underbrace{\frac{1}{2}R_{ab}{}^{kl}\,\xi^a\xi^b}_{\text{curvature 2-form}}\Big) \end{cases} &= \begin{cases} 0, \quad n=2m+1 \\ \displaystyle\int_M \underbrace{\operatorname{Pf}\left(\frac{1}{2\pi}\frac{1}{2}R_{ab}{}^{kl}\,dx^a\,dx^b\right)}_{\text{Euler class of } TM}. \end{cases}
\end{aligned} \tag{4.7}$$

This is the Gauss-Bonnet-Chern formula for the Euler characteristic.

We used the well-known formula for the Pfaffian as a Gaussian integral:

$$\int_{\mathbb{R}^{0|n}} D\theta\; e^{-\frac{1}{2}Q^{ab}\,\theta_a\theta_b} = \begin{cases} 0 & \text{for } n=2m+1 \\ \operatorname{Pf} Q & \text{for } n=2m \end{cases}$$

where $Q^{ab} = -Q^{ba}$ (see, e.g., [16]).

Remark 4.1. The fact that the formula for the trace (2.40) contains no $\hbar$ factors is crucial for our calculation. It comes naturally in our symbol calculus. In the calculus for spinor fields [17] the similar cancellation of the powers of $\hbar$ was achieved only by an artifial "asymmetric" convention for Planck constant.

5 Discussion

With very little effort we were able to deduce the Gauss-Bonnet-Chern Theorem from the trace formula (2.40). Though we possess an analogous trace formula (2.42) for the $*$-grading, more subtle analysis of the symbol is needed

in the case of the Hirzebruch signature operator, because this formula contains $\hbar^{-1}$.

We can also discuss the results with the relation to the problem of quantization of differential algebras, in particular the algebras of forms. In a general setting (cp. [18]), we consider deformation of commutative multiplication in an algebra A, of the form

$$f *_\hbar g = fg + (-i\hbar)\,\{f,g\} + (-i\hbar)^2\ B_2(f,g) + \dots\,, \tag{5.1}$$

starting from some Poisson bracket, and of the differential in A, of the form

$$d_\hbar f = df + (-i\hbar)\,d_1 f + (-i\hbar)^2\,d_2 f + \dots\,, \tag{5.2}$$

all this being considered up to "gauge equivalences" $f \mapsto f + (-i\hbar)\,a_1(f) + \dots$. (Here d_k and a_k are differential operators on A, and B_k are bidifferential operators.) In the first order the derivation property for $d_\hbar$ is equivalent to

$$d\{f,g\} - \{df,g\} - (-1)^{\tilde f}\{f,dg\} = -\Big(d_1(fg) - d_1 f\,g - (-1)^{\tilde f} f\,d_1 g\Big) \tag{5.3}$$

The obstruction to killing d_1 or, more generally, the first nonvanishing term d_n in the expansion (5.2) , $n \geqslant 1$, by a gauge transformation is the cohomology class $[d_n]$ in the complex of operators on A with the differential $\operatorname{ad} d = [d,\ \ ]$.

Consider the case of forms. We hope to analyze the general picture elsewhere, and here we shall make just a few remarks. First, there is a question whether the deformation should respect the $\mathbb{Z}$-grading or just $\mathbb{Z}_2$-grading (parity). For the former option, the induced Poisson bracket should be also $\mathbb{Z}$-graded, for example the bracket of 1-forms should be 2-form (and not a function, for example). This is highly unlikely. Poisson structure obtained above has the canonical brackets

$$\{p_a, x^b\} = \delta_a{}^b, \quad \{\theta_a, \xi^b\} = \delta_a{}^b \tag{5.4}$$

in the coordinates $(x, p, \xi = dx, \theta)$ and is more complicated in the coordinates (x, p, dx, dp). This structure respects parity but by no means $\mathbb{Z}$-grading. The second question concerns the relation between the bracket and d. An easy computation produces the following formulas for the action of d:

$$\begin{aligned}
dx^a &= \xi^a \\
dp_a &= \Gamma^c_{ba}\,\xi^b p_c + \theta_a \\
d\xi^a &= 0 \\
d\theta_a &= \Gamma^c_{ba}\,\xi^b\theta_c - \frac{1}{2}R_{kla}{}^c\xi^k\xi^l p_c,
\end{aligned}$$

which resemble Cartan's equations of structure. Direct check shows that d does not respect the brackets (5.4). (We omit the explicit formulas for $d\{f,g\} - \{df,g\} - (-1)^{\tilde{f}}\{f,dg\}$, which contain both the Christoffel symbol and the Riemann tensor.) It's not surprising, of course, because in our quantization we did not care about d. The open questions are, if d can be "adjusted" by additional terms to satisfy more general condition (5.3), or if some other quantization for $\Omega(T^*M)$ can be found, which would better fit into the picture incorporating d.

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